ISSN: 2277-9655
(I2OR), Publication Impact Factor: 3.785

IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

A COMMON FIXED POINT THEOREM ON FUZZY 3-METRIC SPACES

Mahesh Kumar Verma*, D.P. Sahu
*Ph.D. Scholar, Department of Mathematics, Dr. C.V. Raman University Bilaspur (C.G.)
Retired professor, Department of Mathematics, Govt. Science College (C.G.)

DOI: 10.5281/zenodo.46619

ABSTRACT

In this paper, we prove a common fixed point theorem for four mappings on fuzzy 3-metric spaces. Our result is an extension of results of S. H. Cho [2] to fuzzy 2-metric spaces. Also, it is a generalization of a result of S. Sharma [11].

INTRODUCTION

The concept of fuzzy sets was introduced by L. A. Zadeh [13] in1965. To use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and applications. With the concept of fuzzy sets, the fuzzy metric space was introduced by I. Kramosil and J. Michalek [8] in 1975. M. Grabiec [5] proved the contraction principle in fuzzy metric spaces in 1988. Moreover, A. George and P. Veeramani [4] modified the notion of fuzzy metric spaces with the help of *t*-norms in 1994. GÄahler [3] investigated 2-metric spaces in a series of his papers. Sharma, Sharma and Iseki [12] investigated, for the first time, contraction type mappings in 2-metric spaces. Many authors have studied common fixed point theorems in fuzzy metric spaces. Some of interesting papers are Y. J. Cho [1], George and Veeramani [4], Grabiec [5], Kramosil and Michalek [8] and S. Sharma [11]. S. H. Cho [2] proved a common fixed point theorem for four mappings in fuzzy metric spaces and S. Sharma [11] proved a common fixed point theorem for three mappings in fuzzy 2-metric spaces. In this paper we prove a common fixed point theorem for four mappings in fuzzy 2-metric spaces. Our theorem is an extension of results of S. H. Cho [2] to fuzzy 3-metric spaces. And also, it is a generalization of result of and S. Sharma [11].

PRELIMINARIES

Now we begin with some definitions:

Definition (2 A): A binary operation *: [0, 1] x [0,1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 * d \geq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$for all a_1, a_2 , b_1, b_2, c_1, c_2 and $d_1, d_2$are in [0,1].

Definition (2 B): The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in $X^4 \times [0, \infty)$ satisfying the followings

$(FM'' - 1)$: $M(x, y, z, w, 0) = 0$

$(FM'' - 2)$: $M(x, y, z, w, t) = 1, \forall\, t \succ 0$

$(FM'' - 3)$: $M(x, y, z, w, t) = M(x, w, z, y, t) = M(z, w, x, y, t) = \cdots$ $(FM'' - 4)$: $M(x, y, z, w, t_1 + t_2 + t_3) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(x, y, z, w, t_4)$

Definition (2 C): Let (X, M, *) be a fuzzy 3-matric space. A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$,

$$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1, for\ all\ a, b\ \in\ X\ and\ t \succ 0$$

A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1, for\ all\ a, b\ \in X\ and\ t, p \succ 0$$

A fuzzy 3-matric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2 D): A function M is continuous in fuzzy 3-metric space, iff whenever for all $a\varepsilon X$ and $t > 0$.

$$x_n \to x, y_n \to y, then \lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, t), \forall\, a, b \in X\ and\ t \succ 0$$

Definition (2 E): Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff $M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t), \forall u, a, b \in X \text{ and } t \succ 0$.

Definition (2 F): Self mappings A and B of a fuzzy 3 metric space $(X.M.\Delta)$ is said to be compatible, if $\lim_{n\to\infty} M(ABx_n, BAx_n, a, t) = 1$ for all $a \in X$ and $t \succ 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

MAIN RESULT

Lemma 3.1. $M(x, y, z, w, .)$ is non-decreasing for all $x, y, z, w \in X$.

Proof. Let $s, t > 0$ be any points such that $t > s$. Then $t = s + \frac{t-s}{2} + \frac{t-s}{2}$.

Hence we have

$$M(x,y,z,w,t) = M\left(x,y,z,w,s + \frac{t-s}{2} + \frac{t-s}{2}\right) \geq \Delta M(x,y,z,w,s), M\left(x,y,z,w,\frac{t-s}{2}\right), M(x,y,z,w, \frac{t-s}{2}) = M(x,y,z,w,s)$$

$$Thus, M(x,y,z,w,t) > M(x,y,z,w,s)$$

From now on, let (X, M, Δ) be a fuzzy $2-$metric space with the following condition : $\lim_{t\to\infty} M(x, y, z, w, t) = 1$ for all $x, y, z, w \in X$.

Lemma 3.2. *Let (X, M, Δ) be a fuzzy 3-metric space. If there exists $q \in (0,1)$ such that $M(x, y, z, w, qt + 0) \geq M(x, y, z, w, t)$ for all $x, y, z, w \in X$.*
with $w \neq x, w = y, w \neq z$ and $t > 0$ then $x = y = z$.

Proof. Since $M(x, y, z, w, t) \geq M(x, y, z, w, qt + 0) \geq M(x, y, z, w, t)$ for all $t > 0$, $M(x, y, z, w, .)$ is constant. Since $\lim_{t\to\infty} M(x, y, z, w, t) = 1$, for all $t > 0$.Hence, $x = y = z$.because $w \neq x, w = y, w \neq z$.

Lemma 3.3. *Let (X, M, Δ) be a fuzzy 3-metric space, and let*
$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} u_n = u, \lim_{n\to\infty} v_n = v$*,Then the followings are satisfied .*
(1) $lim\ inf\ M(x_n, y_n, a, b, t) \geq M(x, y, a, b, t) for\ all\ a, b \in X\ and\ t \geq 0.$
(2) $M(x, y, a, b, t + 0) \geq lim\ supM(x_n, y_n, a, b, t)\ a, b \in X\ and\ t > 0.$

Proof. (1) For all $a, b \in X$ and $t > 0$; we have

$$M(x_n, y_n, a, b, t) \geq \Delta(M(x_n, x, a, b, t), M(x_n, y_n, x, y, t); M(y_n, x, a, b, t))$$
$$\geq \Delta(M(x_n, x, a, b, t), M(x_n, x, y_n, a, t), M(y_n, y, a, b, t), M(x, y, a, b, t), M(y_n, x, y, t))$$

which implies
$lim\ inf\ M(x_n, y_n, a, b, t) \geq \Delta(1, 1, 1, M(x, y, a, b, t), 1) = M(x, y, a, b, t)$
for all $a, b \in X$ and $t > 0$.
(2) Let $\varepsilon > 0$ be given. For all $a, b \in X$ and $t > 0$, we have

$$M(x, y, a, b, t + 2\varepsilon) \geq \Delta\left(M\left(x, x_n, a, b, \frac{\varepsilon}{2}\right), M(x_n, y, a, b, t + \varepsilon), M\left(x, y, x_n, a, \frac{\varepsilon}{2}\right)\right)$$

$$\geq \Delta\left(M\left(x_n, x, a, b, \frac{\varepsilon}{2}\right), M\left(x_n, x, y, a, \frac{\varepsilon}{2}\right), M(x_n, y_n, a, b, t), M\left(x_n, y, y_n, a, \frac{\varepsilon}{2}\right), M\left(y_n, y, a, b, \frac{\varepsilon}{2}\right)\right)$$

which implies $M(x, y, a, b, t + 2\varepsilon) \geq \lim supM(x_n, y_n, a, b, t)$. Letting $\varepsilon \to 0$ in the above inequality, we have $M(x, y, a, b, t + 0) \geq \lim supM(x_n, y_n, a, b, t)$.
Note that for all $a, b \in X$ and $t > 0$; in general the inequality
$M(x, y, a, b, t) \geq \lim supM(x_n, y_n, a, b, t)$.is not true, because $M(x, y, z, w, .)$ is left continuous (in general, not right continuous).

Lemma 3.4. *Let (X, M, Δ) be a fuzzy 3-metric space and let A and B be continuous self mappings of X and [A;B] be compatible. Let x_n be a sequence in X such that $Ax_n \to Az$and $Bx_n \to Bz$ Then $ABx_n \to Az$.*

Proof. Since *A,B* are continuous maps, $ABx_n \to Az$, $BAx_n \to Bz$ and so, $M\left(ABx_n, Az, a, b, \frac{t}{3}\right)$and $M\left(BAx_n, Bz, a, b, \frac{t}{3}\right) \to 1$ for all $a, b \in X$ and $t > 0$: Since the pair [*A*,*B*] is compatible, $\left(BAx_n, AB, a, b, \frac{t}{3}\right) \to 1$ for all $a, b \in X$ and $t > 0$: Thus, $M(ABx_n, Bz, a, t) \geq \Delta\left(M\left(ABx_n, Bz, BAx_n, a, \frac{t}{3}\right), M\left(ABx_n, BAx_n, a, b, \frac{t}{3}\right), M\left(BAx_n, Bz, a, b, \frac{t}{3}\right)\right)$
$\geq \Delta\left(M\left(BAx_n, Bz, ABx_n, a, \frac{t}{3}\right), M\left(BAx_n, ABx_n, a, b, \frac{t}{3}\right), M(BAx_n, Bz, a, b, \frac{t}{3})\right) \to 1$ $a, b \in X$ and $t > 0$. Hence, $ABx_n \to Bz$.

Theorem 3.5. *Let (X, M, Δ) be a complete fuzzy 2-metric space with continuous t-norm Δ of H-type, and let S and T be continuous self mappings of X. Then S and T have a unique common fixed point in X if and only if there exist two self mappings A,B of X satisfying*

(1) $AX \subseteq TX, BX \subseteq SX$
(2) *the pair* $[A, S]$ *and* $[B, T]$ *are compatible,*
(3) *there exists* $q \in (0,1)$ *such that for every* $x, y, a, b \in X$ *and* $t > 0$,

$$M(Ax, By, a, b, qt) \geq \{minM(Sx, Ty, a, b, t), M(Ax, Sx, a, b, t), M(By, Ty, a, b, t), M(Ax, Ty, a, b, t)\}. \quad (3.0)$$

Indeed, A,B, S and T have a unique common fixed point in X.

Proof. Suppose that *S* and *T* have a (unique) common fixed point, say $z \in X$. Define $A: X \to X$ by $Ax = z$ for all $x \in X$, and $B: X \to X$ by $Bx = z$ for all $x \in X$. Then one can see that (1)- (3) are satisieed. Conversely, assume that there exist two self mappings *A*,*B* of *X* satisfying conditions (1)- (3). From condition (1) we can construct two sequences $\{x_n\}$and $\{y_n\}$ of *X* such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1,2, \dots \dots$. Putting $x = x_{2n}$ *and* $y = x_{2n+1}$ in (3.0), we have that for all $a \in X$ *and* $t > 0$. In general, we obtain that for all $a \in X, t > 0$ and $n = 1,2, \dots \dots$
$M(y_n, y_{n+1}, a, b, qt), M(y_n, 1, y_n, a, b, t)$. Thus, for all $a \in X, t > 0$ and $n = 1,2, \dots \dots$

$$M(y_n, y_{n+1}, a, b, t) \geq M\left(y_0, y_1, a, b, \frac{q}{t^n}\right). \quad (3.1)$$

We now show that $\{y_n\}$ is a Cauchy sequence in *X*.
Let $\varepsilon \in (0,1)$ be given. Since the *t*-norm Δ is of *H*-type, there exists $\lambda \in (0,1)$such that for all $m, n \in N$ with $m > n$ $\Delta^{2^{m-n}}(1-\lambda) > (1-\varepsilon)$. (3.2)
Since $\lim_{n\to\infty} M\left(y_0, y_1, a, b, \frac{q}{t^n}\right) = 1$, there exists $n_0 \in N$ such that for all $a, b \in X$ and $t > 0$ with $\lim_{n\to\infty} M\left(y_0, y_1, a, b, \frac{q}{t^n}\right) = 1$, for all n , n_0 From(3.1) we have that for all $a, b \in X$ and $t > 0$,

$$M(y_n, y_{n+1,} a, b, t) > 1 - \lambda, for\ all\ n\ , n_0.$$

Let $m > n \geq n_0$. Then for all $a, b \in X$ and $t > 0$ we have

$$M(y_m, y_n, a, b, t) \geq \Delta(M(y_{n+1}, y_n, a, b, 3^{-1}t), \Delta(M(y_{n+1}, y_n, y_m, a, 3^{-1}t), M(y_{n+1}, y_m, a, b, 3^{-1}t)))$$
$$\geq \Delta\left(\Delta^2((1-\lambda), M(y_{n+1}, y_m, a, b, 3^{-1}t))\right) \dots \dots \dots \dots \dots \dots \dots \dots (3:3)$$

Since $M(y_{n+1}, y_n, a, b, 3^{-2}t) \geq \Delta(M(y_{n+2}, y_{n+1}, a, b, 3^{-2}t), \Delta M(y_{n+2}, y_{n+1}, y_m, a, 3^{-2}t), M(y_{n+2}, y_m, a, b, 3^{-2}t))$. from (3.3) we get

$$M(y_m, y_n, a, b, t) \geq \Delta\left(\Delta^{2^2}(1-\lambda), M(y_{n+2}, y_m, a, b, 3^{-2}t)\right).$$

Inductively, we obtain

$$M(y_m, y_n, a, b, t) \geq \Delta(\Delta^{2m-n}(1-\lambda), M(y_m, y_n, a, b, 3^{n-m}t)) = \Delta^{2m-n}(1-\lambda) \dots \dots \dots (3.4)$$

From (3.2) and (3.4) we get for all $a, b \in X$ and $t > 0$ $M(y_m, y_n, a, b, t) > 1 - \varepsilon$ for $m > n \geq n_{0.}$ Thus $\{y_n\}$ is a Cauchy sequence.
It follows from completeness of *X* that there exists $z \in X$ such that
$\lim_{n\to\infty} y_n = z$. Hence $\lim_{n\to\infty} y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} = z$.
From Lemma 3.4, $ASx_{2n-1} = Sz$ *and* $BTx_{2n-1} = Tz \dots \dots \dots \dots \dots (3.5)$
Meanwhile, for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$, and $t > 0$

$$M(ASx_{2n+1}, BTx_{2n+1}, a, b.qt)$$
$$\geq min\begin{Bmatrix} M(SSSx_{2n+1}, TTSx_{2n+1}, a, b, t), M(ASSx_{2n+1}, SSSx_{2n+1}, a, b, t), M(BTx_{2n+1}, TTx_{2n+1}, a, b, t), \\ M(ASx_{2n+1}, TTx_{2n+1}, a, b, t) \end{Bmatrix}.$$

Taking limit as $n \to \infty$, and using (3:5), and Lemma 3.5, we have for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$, and $t > 0$

$$M(Sz, Tz, a, b, qt + 0)$$
$$\geq minM(Sz, Tz, a, b, t), M(Sz, Sz, a, b, t), M(Tz, Tz, a, b, t), M(Sz, Tz, a, b, t), M(Sz, Tz, a, b, t).$$

By Lemma 3.2, we have $Sz = Tz$ ... ... (3.6)

From (3.0) we get for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$, and $t > 0$

$$M(Az, BTx_{2n+1}, a, b, qt)$$
$$\geq min\{M(Sz, TTx_{2n+1}, a, t), M(Az, Sz, a, b, t), M(BTx_{2n+1}, TTx_{2n+1}, a, b, t), M(Az, TTxx_{2n+1}, a, b, t)\}$$

Taking limit as $n \to \infty$, and using (3:5),(3:6) and Lemma 3.3,

$$M(Az, Tz, a, b, qt + 0)$$
$$\geq min\{M(Sz, Tz, a, b, t), M(Az, Sz, a, b, t), M(Tz, Tz, a, b, t), M(Az, Tz, a, b, t)\}$$
$$\geq M(Az, Tz, a, b, t).$$

By Lemma 3.2, $Az = Tz$ ... ... (3.7)

and for all $a \in X$ with $a \neq Sz$ and $a \neq Tz$, and $t > 0$

$$M(Az, BTx_{2n+1}, a, b, qt)$$
$$\geq min\{M(Sz, Tz, a, b, t), M(Az, Sz, a, b, t), M(Bz, Tz, a, b, t), M(Az, Tz, a, b, t)\}$$
$$\geq min\{M(Tz, Tz, a, b, t), M(Tz, Tz, a, b, t), M(Bz, Az, a, b, t), M(Tz, Tz, a, b, t)\}$$
$$\geq M(Az, Bz, a, b, t).$$

By Lemma 3.2, $Az = Bz$ ... ... (3.8)

It follows that $Az = Bz = Sz = Tz$.

For all $a \in X$ with $a \neq Bz$ and $a \neq z$, and $t > 0$

$$M(Ax_{2n}, Bz, a, b, qt)$$
$$\geq min\{M(Sx_{2n}, Tz, a, b, t), M(Ax_{2n}, Sx_{2n}, a, b, t), M(Bz, Tz, a, b, t), M(Ax_{2n}, Tz, a, b, t)\}$$

Taking limit as $n \to \infty$, and using (3.5), and Lemma 3.3, we have for all $a \in X$ with with $a \neq Bz$ and $a \neq z$, and $t > 0$

$$M(z, Bz, a, b, qt + 0)$$
$$\geq min\{M(z, Tz, a, b, t), M(z, z, a, b, t), M(Bz, Bz, a, b, t), M(z, Tz, a, b, t)\} \geq M(z, Tz, a, b, t) \geq M(z, Bz, a, b, t)$$

and so we have $M(z, Bz, a, b, qt) \geq M(z, Bz, a, b, t)$, and hence $Bz = z$.

Thus $z = Az = Bz = Sz = Tz$, and so z is a common fixed point of A,B, S and T.

For uniqueness, let w be another common ¯xed point of A, B, S and T.

Then, for all $a \in X$ with with $a \neq z$ and $a \neq w$, and $t > 0$

$$M(z, w, a, b, qt) = M(Az, Bw, a, b, qt)$$
$$\geq min\{M(Sz, Tw, a, b, t), M(Az, Sz, a, b, t), M(Bw, Tw, a, b, t), M(Az, Tw, a, b, t)\}$$
$$min\{M(z, w, a, b, t), M(z, z, a, b, t), M(w, w, a, b, t), M(z, w, a, b, t)\}$$
$$\geq M(z, w, a, b, t).$$

which implies that $M(z, w, a, b, qt) \geq M(z, w, a, b, t)$ and hence $z = w$.

This complete the proof of Theorem.

REFERENCES

1. Y. J. Cho, *Fixed points in fuzzy metric space*, J. Fuzzy Math. 5 (1997), no. 4, 949-962.
2. S. H. Cho, *On common ¯xed points in fuzzy metric spaces*, Intrnational Mathematical Forum 1 (2006), no. 10, 471-479.
3. S. GÄahler, 2-*metrische Raume and ihre topologische structure*, Math. Nachr. 26 (1983), 15-148.
4. A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems 64 (1994), 395-399.
5. M. Grabiec, *Fixed points in fuzzy metric space*, Fuzzy Sets and Systems 27 (1988), 385-389.
6. O. Hadzic, E. Pap, *Fixed point theory in probabilistic metric spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
7. E. P. Klement, R. Mesiar and E. Pap, *Triangular Norm*, Kluwer Academic Publishers, Trens 8.

8. I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica 11 (1975), 326-334.
9. K. P. R. Rao, G. N. V. Kishore, T. Ranga Rao, *Weakly f-compatible pair* (*f*, *g*) *and common fixed point theorems in fuzzy metric spaces*, Mathematical Sciences 2 (2008), no. 3, 293-308.
10. B. Schweizer, A. Sklar, *Statistical metric spaces*, Paci¯c J. Math. 10 (1960), 313-334.
11. S. Sharma, *On fuzzy metric spaces*, Southeast Asian Bull. of Math. 26 (2002), no. 1, 133-145. 656 Jinkyu Han
12. K. Iseki, P. L. Sharma, B. K. Sharma, *Contractive type mapping on* 2-*metric space*, Math. Japonica 21 (1976), 67-70.
13. L. A. Zadeh, *Fuzzy Sets*, Inform. and Control 8 (1965), 338-353.